MATH1520AB 2021-22 Tutorial 1 (week 2)

1. Let $X = \{1, \pi, \{0\}, \{e, \{1, \pi\}\}\}$. Which of the following are true? Justify your answer.

   (a) $\{\{e, \{1, \pi\}\}\} \in X$

   (b) $\{1, \pi, \{0\}, \{e, \{1, \pi\}\}\} \in X$

   (c) $\{0\} \subseteq X$

   (d) $\{1\} \subseteq X$

   (e) $\{\{0\}\} \subseteq X$

   (f) $\{\{\{0\}\}\} \subseteq X$

   Answer.

   (a) No. The elements of X are $1, \pi, \{0\}, \{e, \{1, \pi\}\}$ and $\{\{e, \{1, \pi\}\}\}$ is not one of them.

   (b) No. Same reason as above.

   (c) No. Since 0 is not an element of X, $\{0\}$ is not a subset of X.

   (d) Yes. The only element in $\{1\}$ is 1 which is an element of X.

   (e) Yes. The only element in $\{\{0\}\}$ is $\{0\}$ which is an element of X.

   (f) No. The only element in $\{\{\{0\}\}\}$ is $\{\{0\}\}$ which is not in X.

2. Let $X = \{A, B, C, D\}$. List all the subsets of X. (Hint: Don't forget the empty set $\emptyset$.)

   Answer.

   $\emptyset, \{A\}, \{B\}, \{C\}, \{D\}, \{A, B\}, \{A, C\}, \{A, D\}, \{B, C\}, \{B, D\}, \{C, D\},$
   $\{A, B, C\}, \{A, B, D\}, \{A, C, D\}, \{B, C, D\}, \{A, B, C, D\}$

3. Determine the natural domain of the following functions.

   (a) $f(x) = \sqrt{-x^2 + 13x - 42}$

   (b) $f(x) = \dfrac{1}{\ln(\sqrt{5 - x})}$

   Answer.

   (a) $f(x) = \sqrt{-x^2 + 13x - 42}$
   $\Rightarrow -x^2 + 13x - 42 \geq 0$
   $\Rightarrow x^2 - 13x + 42 \leq 0$
   $\Rightarrow (x - 6)(x - 7) \leq 0$
   $\Rightarrow 6 \leq x \leq 7$
   Hence, the natural domain is $(6, 7)$.

   (b) $f(x) = \dfrac{1}{\ln(\sqrt{5 - x})}$
   For ln to be well-defined, $\sqrt{5 - x} > 0 \Rightarrow 5 - x > 0 \iff x < 5$
   Also, $\ln(\sqrt{5 - x}) \neq 0 \iff \sqrt{5 - x} \neq 1 \iff 5 - x \neq 1 \iff x \neq 4$
   Hence, the natural domain is $(-\infty, 5) \setminus \{4\} = (-\infty, 4) \cup (4, 5)$.

4. Determine the range of $f : \mathbb{R} \to \mathbb{R}$, $f(x) = |4 - x| - |x - 2|$.

   Answer.

   $$f(x) = \begin{cases} (4 - x) + (x - 2) = 2, & x < 2 \\ (4 - x) - (x - 2) = -2x + 6, & 2 \leq x < 4 \\ -(4 - x) - (x - 2) = -2, & x > 4 \end{cases}$$

   Therefore, the range of f is $[-2, 2]$.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = mx + c$ where m and c are constants. If $f(f(f(x))) = 27x + 26$, find the value of m and c.

   Answer.

   $$f(f(f(x))) = 27x + 26$$
   $$m(m(mx + c) + c) + c = 27x + 26$$
   $$m^3 x + m^2 c + mc + c = 27x + 26$$

   This implies $m^3 = 27$ and $m^2 c + mc + c = 26$.

   So, $m = 3$. We have $9c + 3c + c = 26$ and $c = 2$.